Deconfinement Transition and Flux-String Models

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Abstract

Flux-string models can be used to study the deconfining phase transition. In this note, we study the models proposed by Patel. We also study the large N_c limits of Patel's model. To discuss the validity of the mean field theory results, the one-loop Coleman-Weinberg effective potential is calculated for $N_c = 3$. We argue that the quantum corrections vanish at large N_c when the energy of the so-called baryonic vertices scale with N_c.

I. INTRODUCTION

The structure of the quantum chromodynamics (QCD) the vacuum has been a subject of intense investigations over the last two decades [1], as qualitative understanding of hadron spectroscopy depends on the vacuum configuration. Since the QCD vacuum confines color degrees of freedom, it is highly non-trivial compared to the vacuum for QED. However, to address this issue in a sensible way from the fundamentals of QCD is a rather challenging task as the theory becomes strongly non-perturbative in the infrared region.

It is customary to use effective Lagrangians in the low energy domain to describe the hadronic dynamics using the global symmetries of the theory. However, these models are

inadequate for studying the vacuum structure since the order parameters - the mesonic fields, are color singlets.

Therefore, one has to take a path somewhere in between, explicitly keeping some of the colored degrees of freedom, like quarks, while truncating the other degrees of freedom, namely the gluons. These truncations are performed in various disguises - via potentials, bags, strings etc. One can also argue in favor of the roles played by topologically non-trivial classical solutions like instantons and subsequently perform a semiclassical analysis [2] around these solutions.

An economical picture for confinement is provided by the Meissner effect in type-II superconductors. It is argued that the behavior of QCD vacuum is dual to the type-II superconductors [3,4]. In this picture, the chromoelectric field is screened in the "dual superconductor" just like magnetic field in ordinary superconductors. Therefore the chromoelectric field between a quark and an anti-quark gets squeezed in a tube-shaped region like the Abrikosov flux-tubes in type-II superconductors. This stringy picture for mesons dates back to the Nielsen-Olesen model [5], where the existence of a scalar field is crucial. But, apparently in QCD no such scalars exist. According to 't Hooft the role of the scalars can be mocked up in QCD via a suitable gauge choice and the so-called "Abelian Projection" [6]. Evidences for this type of string models come from the Regge behavior [7] and the hadronization processes in high energy hadronic reactions where it gives a rather consistent picture [8].

It is possible to incorporate baryons in this picture as well [9]. To do so, one has to include special configurations called the "Y-vertices", which allows N_c colored quarks to bind and hence form baryons. These vertices are exclusive to the models with $N_c > 2$ and the flux-strings can form branches in these cases. Evidence for appearance of such objects in the magnetic condensation picture has also been argued by using Makeenko-Migdal equations [10]. It has also been argued that these vertices are gluonic in nature and trace the baryon number [11]. One is allowed to associate some energy with these vertices which is not associated with the flux-tube itself.

On the other hand, at high temperatures it has been argued from general principles [12,13] that a deconfining phase in QCD exists. An important issue is then the nature of the phase transition. The order of phase transition can be found by using universality arguments and looking at appropriate spin models which belong to the same universality class as QCD [14] for various color groups. In these spin models, the phase transition proceeds via the formation of domains at lower temperatures by cooling the system from a high temperature phase where one is guaranteed a disordered state which can be understood as the deconfined phase.

One can use the flux-string models to study the same problem, though here one adopts a bottom-up approach - i.e. we start at lower temperature in the confining phase and then raise the temperature of the system. This forces the flux-string to fluctuate and at high temperatures the string will elongate and make a criss-crossed mesh. At the deconfining point this net will permeate the whole space. This way of addressing the deconfinement transition in a flux-string model was studied by Patel [15,16].

In the original work of Patel [16], a flux-string model was proposed on the lattice where the strings lived on the links and the quarks and the baryonic vertices were defined on the lattice sites subject to the Gauss law for the conservation of $U(1)$ color charges modulo N_c. The model is simple enough to be studied analytically yet it retains much of the relevant physics in a model. Patel used mean field theory to evaluate the free energy of the system. In this treatment, the system showed phase-transition of second order for $N_c = 2$ and of first order when $N_c = 3$ provided one takes into account the baryonic "Y-vertices". This result qualitatively agrees with the result found by looking at the center of the gauge group [14]. The difference between $N_c = 2$ and $N_c = 3$ arises from the presence of the so-called "Y-vertices" in the string picture. In fact, this is consistent with the model of Patel.

In fact, the gluonic vertices resolve the following puzzle. In absence of these vertices and quarks, the phase transition predicted by Patel's model is of second order, independent of the value of N_c. This originates from the fact that the 3D pure XY model has a second order phase transition [17], although some subtleties with this result exist in the literature [18]

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(see also [19]). The universality arguments of [14] on the other hand show that the QCD phase transition for $N_c = 3$ is of first order, even in the case of pure gauge fields. Therefore, to resolve this apparent conflict, it is natural to include these vertices in these models and require them to be gluonic in nature.

At first sight, Patel's results might seem a little unsettling as they rely on a mean field treatment as one is aware of the role of fluctuations in lower dimensions. However, as we will show below, this result is correct provided the results are interpreted in terms of large N_c. This is the issue we would like to address in this paper.

We take the following route. The model that Patel gets on the lattice is the non-linear U(1) σ model in three dimensions with some specific type of symmetry breaking. This model has a physical cutoff built into it, namely the thickness of the string. We evaluate the one-loop effective potential for this model in the continuum. We find that there are contributions which are non-analytic in the coupling constants. Being non-analytic they can't be absorbed by any finite choice of counter-terms and hence must be included in the initial theory. Therefore, we include the non-analytic terms in the free-energy expression and hence perform an "improved" mean-field theory calculation.

The paper is arranged as below. In section 2, we discuss the behavior of these baryonic vertices with increasing N_c. Next in section 3, we discuss the flux-string model for the simplest case of $N_c = 2$ where all hadrons can be represented simply as unbranched flux tubes and correspondingly the problem can be mapped into a non-backtracking random walk on a cubic lattice. We also review the model with $N_c \geq 3$ where the flux-tubes can branch and this warrants a different treatment from the $N_c = 2$ case. In section 3, we give details of the mean field calculation done by Patel. In section 4,we discuss the naive continuum limit for the model of section 3 and discuss the phase structure using the continuum model. In section 5, we calculate the one-loop effective potential using the Coleman-Weinberg results and find the non-analytic contribution to the tree level effective potential. The non-analytic piece is then included and the mean field free energy is computed for the modified system.

II. BARYONIC VERTICES AT LARGE N_C

The string model can also serve as a model for large N_c QCD. In both models baryons are qualitatively differnt from mesons. In string theory this difference is coded in the N-string vertex. Witten [20] has argued that the mass associated with the vertex increase with N. We will incorporate this behavior in our picture. The flux tube generated in three color QCD has a finite cross sectional area. This area a^2 is expected to shrink as $\frac{a_0^2}{N}$ in the limit as N goes to infinity, where a is the width of the string and a_0 is a constant. Since in the string model of a baryon there are N strings converging on the vertex the total cross sectional area of all the strings is finite. Thus there is finite region surrounding the vertex, of radius $R \sim a_0$, where the strings overlap (see Fig. 1).

It is this region of finite volume which provides the common background interaction for all the quarks (with the strings) that is the essence of Witten's large N picture of the baryons. According to Witten [20] this background interaction is responsible for a contribution of O(N) to the baryon mass. Since, this junction of finite area is present for N_c , we incorporate this contribution as the effective mass v of the N-string junction and thus have $v \to N$ as $N \to \infty$.

It is quite interesting that the above picture for baryons can be merged nicely with the Skyrmion picture for baryons. Since these flux tubes repel each other (like their type-II superconductor counterpart), it is only natural that at large N_c, baryons would look like hedgehogs with quarks stuck at the ends of each "needle". We believe this picture is quite parallel to the case when one can interpolate between the bag or string picture for mesons [21].

III. FLUX TUBE MODELS FOR DECONFINEMENT

A. SU(2) flux-strings : Random Walk

We review the flux-string picture of deconfining transition by describing the simplest case, when the color gauge group is $SU(2)$. For $SU(2)$ gauge theory, flux tubes represent both mesons (made of a quark and an anti-quark) and baryons (made of two quarks). On the lattice, these flux strings can be represented by locally non-backtracking random walks [15]. So, they are different from self-avoiding random walks, as they can cross each other after several steps. The Hamiltonian for a flux tube of length l is given by

$$H = \sigma a l \tag{3.1}$$

and the grand partition function is

$$Z(\beta) = \sum_l e^{-(\beta \sigma l a)} \tag{3.2}$$

For non-back tracking motion on a d-dimensional hypercubic lattice, there are $(2D - 1)$ different directions that the flux tube can move to and hence for a flux tube of length l in lattice units one has $(2D - 1)^l$ different configurations when the ends of the flux tube is allowed to be free. Hence the grand partition function is [1]

$$Z(\beta) = \sum (2D - 1)^l e^{-\beta \sigma l a} = \sum e^{-l[\beta l a - \ln(2d-1)]}. \tag{3.3}$$

The partition function diverges, which is a typical signature for phase transitions, at the temperature -

$$k_B T_c = \frac{\sigma a}{\ln(2d - 1)}. \tag{3.4}$$

In fact, this is a second order phase transition. This model has recently been discussed by Kiskis also [22].

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[1] one can include a chemical potential for quarks with finite masses, though we will ignore it hereafter.

B. Case for $N_c \geq 3$

The more general model proposed by Patel [15,16] can be described as follows. In this model one can include flux-string branchings which are present in models with $N > 2$, and hence this is more general than the model presented above.

As before, we work on a D-dimensional hypercubic lattice. The fluxtubes are described via the the variables $n_{i,\mu}$ living on the μ-th link on the lattice site i which are valued in the set $\{0, \pm 1\}$. Also one has quarks occupying the lattice sites described by the variable $p_i \in \{0, \pm 1\}$ and the baryonic vertices described by the variable $q_i \in \{0, \pm 1\}$. The Hamiltonian describing the system is given by

$$H = \sum_i [\sigma a \sum_\mu n_{i,\mu}^2 + m p_i^2 + v q_i^2] \tag{3.5}$$

where v is the energy associated with each N-legged vertex representing the baryons. However, as in the gauge theory, one also has the Gauss law constraint

$$\sum_\mu (n_{i,\mu} - n_{i-\mu,\mu}) - p_i + N q_i \equiv \alpha_i = 0 \tag{3.6}$$

which we need to implement at each lattice site. Hence the partition function is

$$Z(\sigma a, \beta) = \sum e^{-\beta H} \prod_i \delta_{\alpha_i, 0} \tag{3.7}$$

The partition function can be evaluated by rewriting the delta function as

$$\delta_{\alpha_i, 0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta_i e^{i\alpha_i \theta_i} \tag{3.8}$$

This allows us to express our partition function as

$$Z = \prod_i \int \frac{d\theta_i}{2\pi} \sum_{n_{i,\mu}} e^{-(\beta \sigma a n_{i,\mu}^2 + i n_{i,\mu}(\theta_{i+\mu,\mu} - \theta_{i,\mu}))} \sum_{p_i} (e^{-(\beta m p_i^2 + i p_i \theta_i)}) \sum_{q_i} e^{-(\beta v q_i^2 + i N q_i \theta_i)} \tag{3.9}$$

Carrying out the summation over the variables $n_{i,\mu}, p_i, q_i$ we get

$$Z(\beta) = \prod_i \int \frac{d\theta_i}{2\pi} \prod_\mu (1 + 2e^{-\beta \sigma a} \cos(\theta_{i+\mu} - \theta_i))(1 + 2e^{-\beta m} \cos \theta_i)(1 + 2e^{-\beta v} \cos(N\theta_i)) \tag{3.10}$$

However, we are interested in the limit $\beta \to \infty$ and hence it is expedient to replace terms like $(1 + 2e^{-\beta m} \cos \theta_i)$ by $e^{h \cos \theta_i}$ where $h = 2e^{-\beta m}$. The partition function is then given by

$$Z(\beta) = \int \prod_i d\theta_i e^{J \sum_{\mu,i} \cos(\theta_{i+\mu} - \theta_i) + h \sum_i \cos \theta_i + p \sum_i \cos(N\theta_i)} \tag{3.11}$$

where

$$J = 2e^{-\beta \sigma a} \qquad h = 2e^{-\beta m} \qquad p = 2e^{-\beta v} \tag{3.12}$$

When the baryon vertices are very massive, i.e. $p \to 0$, the expression (3.11) reduces to the partition function of the D dimensional XY model with an external magnetic field.

The mean field free energy can be calculated (as in Appendix 1) and one gets [16],

$$\beta F \leq -[\ln I_0(z) + DJ(\frac{I_1(z)}{I_0(z)})^2 + (h - z)\frac{I_1(z)}{I_1(z)} + p\frac{I_N(z)}{I_0(z)}] \tag{3.13}$$

Let us now deal with the case $N_c = 3$ in detail. For small values of z it is sufficient to study the critical parameters (3.13) using a series expansion in z for the free energy

$$\beta F_{m.f.} = -\frac{hz}{2} + \frac{z^2}{4}(1 - DJ) - \frac{z^3}{48}(p - 3h) - \frac{3z^4}{64}(1 - \frac{4D}{3}J) + O(z^5). \tag{3.14}$$

Using this and assuming small values for p and h one can easily find the critical parameters [16]. When the quarks are infinitely massive (i.e. $h = 0$) one finds

$$1 - DJ_0 = \frac{p^2}{36} \tag{3.15}$$

where J_0 is the critical parameter at the which the system shows phase transition. When $p = 0$, one can see $J_0 = \frac{1}{D}$ and the system shows a *second order* phase transition.

However, when quarks have some finite mass, $h \neq 0$ and the critical parameters are found to be

$$1 - DJ_0 = \frac{p^2}{24} \qquad h_{cr} = \frac{p^3}{216}. \tag{3.16}$$

When $h > h_{cr}$, the system shows no phase transition.

The above mean field treatment shows that the model exhibits a second order phase transition which is the case for $N_c = 2$ and this result seems in agreement with the results

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of [14]. However, as one knows mean field treatment is not reliable in $d > 4$, one should treat this result rather cautiously.

Before we attempt to discuss the stability of the phase structure against quantum fluctuations, it is interesting to discuss the phase transition in this type of model for arbitrary $N_c > 3$.

In the second paper of Patel [16] it was claimed that the order of the phase transition depends on the value of p. When $h = 0$, it was argued that for low values of p the phase transition is of second order and at some high values of p it is of first order. This can be checked explicitly by plotting the mean field free energy given by (3.13)(see Fig. 2,3,B, showing the critical cases). Though this is, in principle, true, we don't consider the result reliable.

The second vacua appears at large values of z and for p near its maximum value of 2. This critical value of p seems to increase as N_c increases (see Figures). On the contrary large N_c argument imply that p actually decreases as N_c increases. Thus for large N_c, implying small p we expect only a second order phase transition. Given a physically reasonable value of p, (for $N_c = 3$), $p < 1$, Patel's model seems to imply a second order phase transition for $N_c \geq 4$, in agreement with [14]. This result also supports the recent arguments of [23] that a second order phase transition for $N_c \geq 4$ would provide insight into the weakly first order QCD deconfinement transition for $N_c = 3$. In our context, $N_c = 3$ is somewhat special.

IV. RELIABILITY OF MEAN FIELD RESULTS

To check validity of the mean field results, we have to compare one-loop contribution to the free energy with the mean field result. However, as it stands the Hamiltonian is awkward for such a perturbative calculation, since the interaction terms are nonpolynomial. The way to get around this is to notice that one can rewrite the Hamiltonian as the nonlinear $O(2) \sim U(1)$ chiral model. Defining $U = e^{i\theta}$

$$- \beta H = \sum_i \frac{1}{2} \left[\sum_\mu J(U_i U_{i+\mu}^\dagger + h.c.) + h(U_i + U_i^\dagger) + p(U_i^N + U_i^{\dagger N}) \right] \tag{4.1}$$

Notice that the quark masses appears like the chiral symmetry breaking term as in the standard chiral Lagrangians but we also have the extra term due to the baryonic vertices . We also have to impose the constraint

$$UU^\dagger = 1 \tag{4.2}$$

At long wavelengths we can replace the first term in equation (4.1) by the derivative so that the Hamiltonian reads

$$- \beta H = \int d^3x \frac{1}{2} \left[-\tilde{J} \nabla U \cdot \nabla U^\dagger + \tilde{h}(U + U^\dagger) + \tilde{p}(U^N + U^{\dagger N}) \right] \tag{4.3}$$

where we have scaled the various coupling constants as follows.

We will be interpreting the above model as an effective theory with a short distance cutoff a_0 which is the thickness of the string. This cutoff will be present as the fluxtube description breaks down at distances shorter than the string thickness. In terms of a_0 one can see $\tilde{p} = \frac{p}{a_0^3}$, $\tilde{h} = \frac{h}{a_0^3}$ and $\tilde{J} = \frac{J}{a_0}$. We will restore this cutoff later.

Even when $\tilde{p}, \tilde{h} = 0$ this model is nontrivial due to its nonlinear structure and shows a first order phase transition which is understood as a vortex loop condensation [24]. This model also has an interesting large N_c behavior as can be understood as follows. The quantum contribution to effective action for this model involves $(N-2)$th order loop and hence has a factor of $\hbar^{N-2}$. As $N_c \to \infty$, this goes to zero and hence the theory is expected to become classical.

Hereafter we will restrict ourselves to the case $\underline{N = 3 \text{ only}}$. Writing

$$U = (\phi + i\psi); \qquad \phi^2 + \psi^2 = 1 \tag{4.4}$$

we get after eliminating ψ

$$\beta H = \int d^3x \left[\frac{1}{2} \tilde{J} \frac{1}{1 - \phi^2} |\nabla \phi|^2 - (\tilde{h} - 3\tilde{p})\phi - 4\tilde{p}\phi^3 \right] \tag{4.5}$$

Note that in terms of variables θ , $\phi = cos\theta$. The potential term

$$V(\phi) \equiv -(\tilde{h} - 3\tilde{p})\phi - 4\tilde{p}\phi^3 \tag{4.6}$$

admits a local minimum only if $3\tilde{p} > \tilde{h}$, which is given by

$$\phi_0 = -\sqrt{\frac{3\tilde{p} - \tilde{h}}{12\tilde{p}}} \equiv -v \tag{4.7}$$

Lets expand the "Hamiltonian" (4.5) about this vacuum,

$$\phi \equiv -v + \sigma \tag{4.8}$$

where σ is the fluctuation about the vacuum v. Then one has

$$\beta H = \int d^3x[-4\tilde{p}v^3 + \frac{J}{2(1 - v^2)}\nabla\sigma \cdot \nabla\sigma - 12\tilde{p}v\sigma^2 - 4\tilde{p}\sigma^3] \tag{4.9}$$

Rescaling the field $\sigma = \sqrt{\frac{9\tilde{p}+\tilde{h}}{12\tilde{p}\tilde{J}}}\chi$ one can rewrite the "Hamiltonian" for the fluctuation (after dropping the constant piece) as

$$\beta H' = \int d^3x[\frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}M^2\chi^2 - \frac{1}{3!}g\chi^3] \tag{4.10}$$

where

$$M^2 \equiv \frac{9\tilde{p} + \tilde{h}}{\tilde{J}}\sqrt{1 - \frac{\tilde{h}}{3\tilde{p}}}$$

$$g \equiv \frac{1}{\sqrt{3\tilde{p}}}(\frac{9\tilde{p} + \tilde{h}}{\tilde{J}})^{\frac{3}{2}} \tag{4.11}$$

Note that when $\tilde{p} = \frac{\tilde{h}}{3}$ the fluctuations become massless.

V. COLEMAN-WEINBERG EFFECTIVE POTENTIAL

Though the nature of phase transition can be found by looking at the classical potential, the presence of fluctuations might change the structure in lower dimensions , i.e. mean field calculations based on Gaussian fixed points are not reliable. To find the deviations, it is customary to look at the one-loop effective potential, as calculated first by Coleman and

Weinberg [25]. Given a classical potential $V(\phi)$, the effective potential up to one-loop is given by

$$V_{one-loop}(\phi) = V(\phi) + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln(G^{-1}(k) + \frac{\partial^2 V(\phi)}{\partial \phi^2}) \tag{5.1}$$

For our case, $V(\phi)$ is as in (4.6) and the inverse propagator $G^{-1}(k)$ is given by $\frac{Jk^2}{1-\phi^2}$ so that one-loop contribution to the the effective potential reads,

$$V_1(\phi) \equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln(\frac{Jk^2}{1-\phi^2} - 24\tilde{p}\phi) \tag{5.2}$$

The finite contribution from this integral can be evaluated by the dimensional regularization method, details of which can be found in the appendix. The answer is

$$-\frac{\hbar}{12\pi}[W(\phi)]^{\frac{3}{2}} \tag{5.3}$$

where

$$W(\phi) \equiv -\frac{24\tilde{p}}{\tilde{J}}\phi(1 - \phi^2) \tag{5.4}$$

Note that the one-loop contribution is real when ϕ is negative. In terms of the original variables namely θ, this one-loop contribution reads

$$-4\sqrt{6}\frac{\hbar}{\pi}(\frac{\tilde{p}}{\tilde{J}})^{\frac{3}{2}} \cos^{\frac{3}{2}}\theta \sin^3\theta \tag{5.5}$$

This expression is non-analytic and hence can not be absorbed by a suitable counter. Hence, we follow the following strategy - namely, include this term in the outset and hence redo the mean field calculation. Basically, we are then working with an effective action which includes the non-analytic term. Restoring the cutoff one sees that the relevant term to add in the lattice theory would be

$$-4\sqrt{6}\frac{\hbar}{\pi}(\frac{p}{Ja_0^2})^{\frac{3}{2}} \cos^{\frac{3}{2}}\theta \sin^3\theta. \tag{5.6}$$

We would like comment on this quantum correction in the large N_c limit. It is known that [26] in large N_c limit the thickness of the strings goes to zero with $a_0 \sim \frac{1}{\sqrt{N_c}}$. So, naively

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the contribution to this term increases with N_c. However, if $v \sim N_c$ then one sees that this correction goes to zero as $N_c \to \infty$. Hence, the theory becomes classical, which is consistent with our naive expectations.

In terms of the lattice theory, the inclusion of this term adds the following term in (A6)

$$- 4\sqrt{6}\frac{\hbar}{\pi}(\frac{p}{Ja_0^2})^{\frac{3}{2}}\langle\cos^{\frac{3}{2}}\theta\sin^3\theta\rangle \tag{5.7}$$

However, this contribution is non-analytic in $z = cos\theta$. Hence, we have to take a branch-cut along the positive z-axis and take the positive root above the cut and the negative root below the cut. Then, one can readily evaluate the above integral, details of which we will relegate to the appendix, as

$$- 8\sqrt{6}\frac{\hbar}{\pi}(\frac{p}{Ja_0^2})^{\frac{3}{2}}\frac{\pi}{I_0(z)}[\frac{5}{2}{}_1F_1(\frac{5}{2},\frac{7}{2};z) - \frac{9}{2}{}_1F_1(\frac{9}{2},\frac{11}{2};z)] \tag{5.8}$$

As a series expansion in z, this reads

$$- \frac{32\sqrt{6}\hbar}{\pi^2}(\frac{p}{Ja_0^2})^{\frac{3}{2}}[0.022 + 0.013z - 0.0013z^2 - 0.0022z^3 - .00188z^4 + O(z^5)] \tag{5.9}$$

this term will now contribute to the mean field free energy given by (3.14). Defining $\lambda \equiv \frac{32\sqrt{6}\hbar}{\pi^2}(\frac{p}{Ja_0^2})^{3/2}$, the total free energy is given by

$$\beta F = -0.022\lambda - z(\frac{h}{2} + .013\lambda) + z^2(\frac{1-3J}{4} + .0013\lambda) - z^3(\frac{p-3h}{48} - .0022\lambda) \tag{5.10}$$
$$- z^4(\frac{3(1-4J)}{64} - .00188\lambda) + \cdots.$$

The behavior is this free energy function is similar to the mean field result and one can show that the phase transition do not change.

CONCLUSIONS

In this paper, we have attempted to incorporate quantum fluctuations in a flux-tube model by looking at a continuum limit of the model. The effect of the quantum fluctuations are interesting though they do not change the nature of the deconfining phase transition.

However, to show this one has to use the fact that the "Y-vertices" has to scale with N_c. Only then is the picture consistent with large N_c. The large N_c limit of Patel's model implies that for $N_c \geq 4$, the phase transition is of second order.

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REFERENCES

[1] E. Shuryak, The QCD vacuum, hadrons and superdense matter, World Scientific, Singapore (1988).

[2] C. G. Callan, R. Dashen and D. Gross, Phys. Rev. **D 17** (1977).

[3] G. 't Hooft, CERN report no. 1902, 1974 (unpublished), in *High Energy Physics*, EPS International Conference, Palermo, 1975, ed. A. Zichichi; S. Mandelstam, Phys. Rept. **23 C** (1976)245; G. Parisi, Phys. Rev. **D 11** (1976)971, V.P. Nair and C. Rosenzweig, Phys. Rev. **D 31**(1984)401 and references therein.

[4] For a review of Dual QCD see, M. Baker, J.S. Ball and F. Zachariasen, Phys. Rept. **209** (1991)73; T. Suzuki, Nucl. Phys. Proc. Supp. **30** (1993)176 and references therein ; K. Yee , Helv. Phys. Acta. **67** (1994)345 (hep-ph/9404363).

[5] H.B. Nielsen and P. Olesen, Nucl. Phys. **B 61** (1973)45. See also, B. Zumino in *Renormalization and Invariance in Quantum Field Theory*, ed. E. Caianiello (Plenum Press, 1974)p. 367.

[6] G. 't Hooft, Nucl. Phys. **B 138** (1978)1 ; *ibid.* **B 153** (1979)141.

[7] See for example P.D.B. Collins , An introduction to Regge theory and high energy physics, Cambridge Univ. Press (1977).

[8] B. Andersson et. al. , Phys. Rept. **93**(1983)33 .

[9] G.C. Rossi and G. Veneziano, Nucl. Phys. **B 123**(1977) 507; G. Veneziano , G.C. Rossi and L. Montanet, Phys. Rept. **63** (1980) 149.

[10] P. Olesen, in *Monopoles in Quantum Field Theory*, ed. N.S. Craigie, P. Goddard and W. Nahm, World Scientific, Singapore (1982), p.315.

[11] D. Kharzeev, Phys. Lett. **378**(1996)238.

[12] A. Polyakov, Phys. Lett. **72 B**(1978)477.

[13] L. Susskind, Phys. Rev. **D 20**(1979)2610.

[14] B. Svetitsky and L. Yaffe, Nucl. Phys. **210** [FS 6] (1982)423. For a review, see B. Svetitsky , Phys. Rept. **132** (1985)1.

[15] A. Patel, Nucl. Phys. **B 243** (1984) 411. See also, F. Takagi, Phys. Rev. **D 34**(1986)1646.

[16] A. Patel, Phys. Lett. **B 139** (1984) 394.

[17] J. Zinn-Justin and B. Le Guillou, Phys. Rev. **B 21** (1978) xxx.

[18] H. Kleinert, Phys. Lett. **93 A** , (1982)86; W. Jahnke and H. Kleinert, Nucl. Phys. **B 270**(1986)399.

[19] M. Stone and P. Thomas, Nucl. Phys. **B 144** (1978) 513.

[20] E. Witten, Nucl. Phys. **B 160** (1979) 57.

[21] V.P. Nair and C. Rosenzweig, Nucl. Phys. **B 250**(1985) 729.

[22] J. Kiskis, Phys. Rev. **D 49** (1994) 2597; *ibid.* **D 51** (1995) 3781 ; Univ. Calif. DAVIS preprint - UCD-95-14 (hep-lat/9510029).

[23] R. D. Pisarski and M. Tytgat, Brookhaven preprint (hep-ph/97022340). See also, M. Teper, Oxford preprints (hep-lat/9701003, hep-lat/9701004).

[24] See for instance H. Kleinert, Gauge fields in Condensed Matter Physics, vol.1 (World Scientific, Singapore,1989).

[25] S. Coleman and E.J. Weinberg, Phys. Rev. **D 7**(1973) 1888.

[26] See for instance, C. Rosenzweig, Phys. Rev. **D 38** (1988)1934.

APPENDIX A: MEAN FIELD CALCULATION

Here we reproduce the Free energy expression for the model using the mean field cal-culation. Remember that our model is defined in three dimensions and hence mean field

calculations are not reliable. Then the partition function can be written as

$$Z_0 = \int \prod_i d\mathbf{S}(\theta)_i e^{J \sum_{\mu,i} \mathbf{S}_i \cdot \mathbf{S}_{i+\mu} + h \sum_i \mathbf{B} \cdot \mathbf{S}_i} \tag{A1}$$

To evaluate the partition function, we now resort to mean-field where the spins are assumed to produce a mean magnetic field z so that the partition function is

$$Z(\beta) = \int \prod_i \frac{d\theta_i}{2\pi} e^{z \cos\theta_i} e^{J \sum_{\mu,i} \cos(\theta_{i+\mu} - \theta_i) + (h-z) \sum_i \cos\theta_i + p \sum_i \cos(N\theta_i)}$$

$$= [I_0(z)]^N \langle e^{J \cos(\theta_{i+\mu} - \theta_i) + h\cos\theta_i + p \cos(N\theta_i)} \rangle_{M.F.} \tag{A2}$$

where $I_N(z)$ is the Bessel function with imaginary arguments of Nth order and $\langle \cdots \rangle_{M.F.}$ is the expectation value evaluated using the mean field weighing factor,i.e.

$$\langle A \rangle_{M.F.} \equiv \frac{\prod_i \int \frac{d\theta_i}{2\pi} A(\theta) e^{z \cos\theta_i}}{\prod_i \int \frac{d\theta_i}{2\pi} e^{z \cos\theta_i}} \tag{A3}$$

Now we use the fact for any convex function one has the following inequality

$$\langle e^A \rangle \geq e^{\langle A \rangle} \tag{A4}$$

Hence the partition function is bounded below by the following quantity

$$Z(\beta) \geq [I_0(z)]^N e^{\langle J \cos(\theta_{i+\mu} - \theta_i) + (h-z) \cos\theta_i + s \cos(N\theta_i) \rangle} \tag{A5}$$

Recall that the free energy expression is given by $F = \lim_{N \to \infty} -\frac{1}{\beta N} \ln Z$ and hence

$$\beta F \leq \beta F_{M.F.} = -\left[\ln[I_0(z)] + \langle J \sum_\mu \cos(\theta_{i+\mu} - \theta_i) + (h - z) \cos\theta_i + s \cos(N\theta_i) \rangle \right] \tag{A6}$$

Now notice that $\langle \sin\theta \rangle = 0$ and thus

$$\beta F \leq -\left[\ln[2\pi I_0(z)] + 3\langle \cos\theta_i \rangle^2 + (h - z)\langle \cos\theta_i \rangle + s\langle \cos N\theta_i \rangle \right] \tag{A7}$$

The various averages can be carried out easily by noting

$$< \cos N\theta > = \frac{\int_0^{2\pi} d\theta e^{\alpha \cos\theta} \cos N\theta}{\int_0^{2\pi} e^{\alpha \cos\theta}}$$

$$= \frac{\int_0^{2\pi} d\theta \sum_{n=-\infty}^{\infty} I_n(\alpha) \cos n\theta \cos N\theta}{\int_0^{2\pi} d\theta \sum_{n=-\infty}^{\infty} I_n(\alpha) \cos n\theta} \tag{A8}$$

which gives us the result ,

$$< \cos N\theta >= \frac{I_N(\alpha)}{I_0(\alpha)}. \tag{A9}$$

Using this is (A8), we get the expression (3.13).

APPENDIX B: THE ONE-LOOP CONTRIBUTION TO THE FREE ENERGY

In this appendix we give the details of our various computations. From (5.2), one can see that thew one-loop contribution can be written as

$$V_1(\phi) = \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} [\ln(k^2 - 24\frac{\tilde{p}}{\tilde{j}}\phi(1-\phi^2)) - \ln(\frac{(1-\phi^2)}{\tilde{j}})]. \tag{B1}$$

The second term in the integrand can be dropped in dimensional regularization. Defining $W(\phi) \equiv -24\frac{\tilde{p}}{\tilde{j}}\phi(1-\phi^2)$, one can find

$$\frac{\partial V_1(\phi)}{\partial \phi} = \frac{\hbar W'(\phi)}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + W(\phi)} = \frac{\hbar W'(\phi)}{16\pi^{3/2}}\Gamma(-\frac{1}{2})\sqrt{W(\phi)} \tag{B2}$$

$$= -\frac{\hbar}{8\pi}\sqrt{W(\phi)}W'(\phi)$$

with $W'(\phi) = \frac{\partial W}{\partial \phi}$. Integrating both sides and dropping an irrelevant constant, we get

$$V_1(\phi) = -\frac{\hbar}{12\pi}[W(\phi)]^{3/2} = -\frac{4\sqrt{6}\hbar}{\pi}(\frac{\tilde{p}}{\tilde{j}})^{3/2}\{\phi(1-\phi)\}^{3/2}. \tag{B3}$$

Putting in $\phi = \cos\theta$ one gets the answer (5.3).

Next we calculate the mean-field contribution due to $V_1(\phi)$. Now,

$$< \cos^{3/2}\theta\sin^3\theta > = \frac{\int_0^{2\pi} d\theta\, e^{z\cos\theta}cos^{3/2}\theta\sin^3\theta}{\int_0^{2\pi} e^{z\cos\theta}} \tag{B4}$$

$$= \frac{1}{2\pi I_0(z)} \sum_{m=0}^{\infty} \frac{z^m}{m!}(\int_0^{2\pi} d\theta \cos^{3/2+m}\theta\sin\theta - \int_0^{2\pi} d\theta \cos^{7/2+m}\theta\sin\theta).$$

The integrals above can be performed by noting that $\sqrt{\cos\theta}$ is not an analytic function and we have to introduce a branch-cut as stated previously. We choose $\sqrt{\cos\theta} = 1$ for $\theta = 0$ and $\sqrt{\cos\theta} = -1$ for $\theta = 2\pi$. Then we can readily evaluate

$$< \cos^{3/2}\theta\sin^3\theta > = \frac{1}{\pi I_0(z)} \sum_{m=0}^{\infty} \frac{z^m}{m!}\{\frac{1}{m+5/2} - \frac{1}{m+9/2}\} \tag{B5}$$

$$= \frac{2}{\pi I_0(z)}[\frac{1}{5}{}_1F_1(5/2, 7/2, z) - \frac{1}{9}{}_1F_1(9/2, 11/2, z)]$$
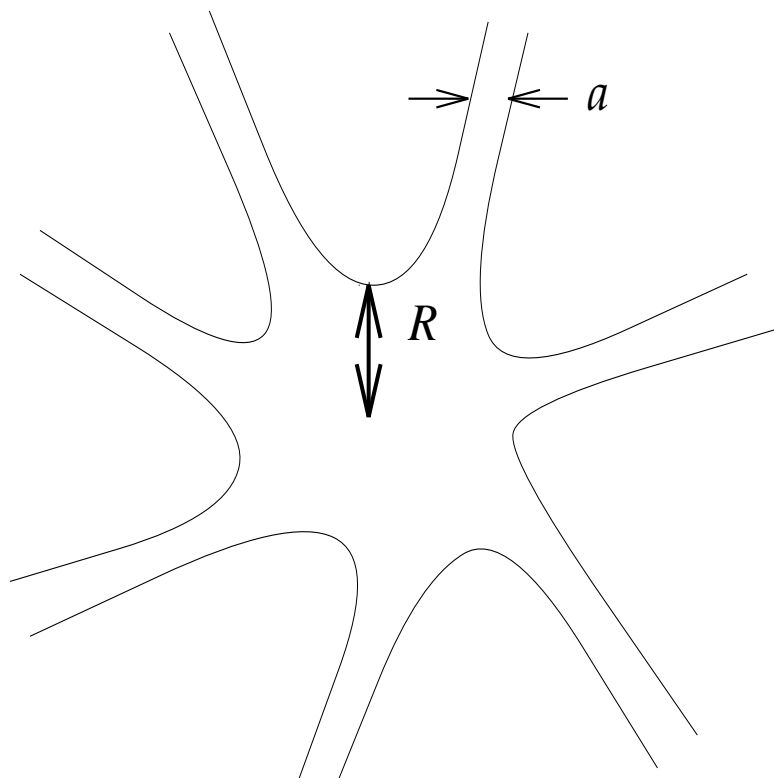
which leads us to (5.8).
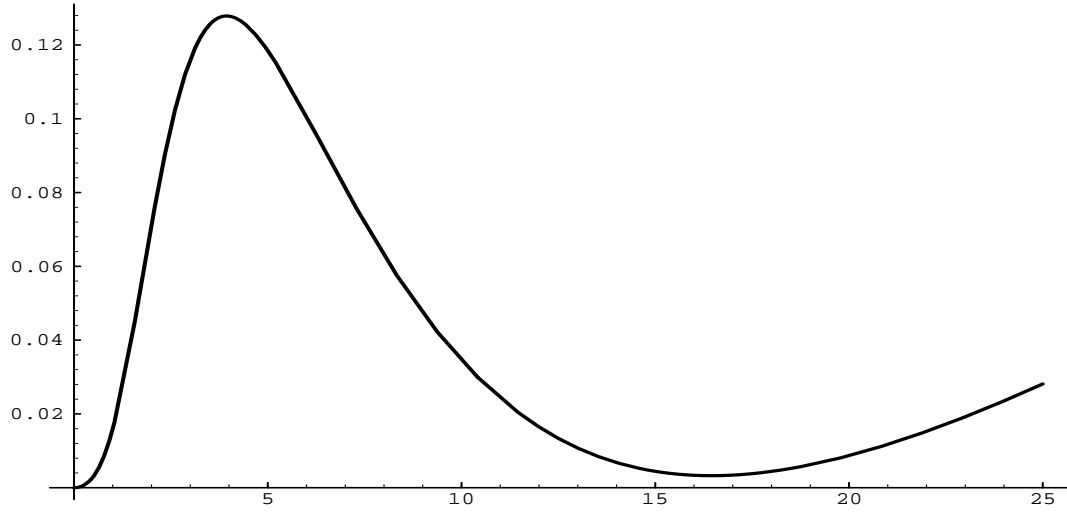
18

FIG. 1. Baryonic vertices with finite volume

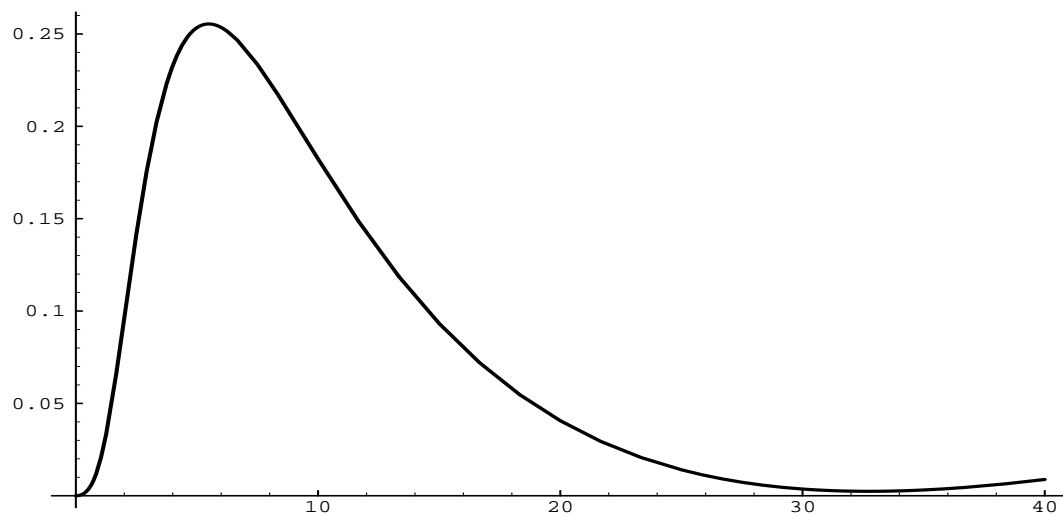FIG. 2. Free Energy as a function of z for $N_c = 4$, $p = 1.48$, $J = .32$

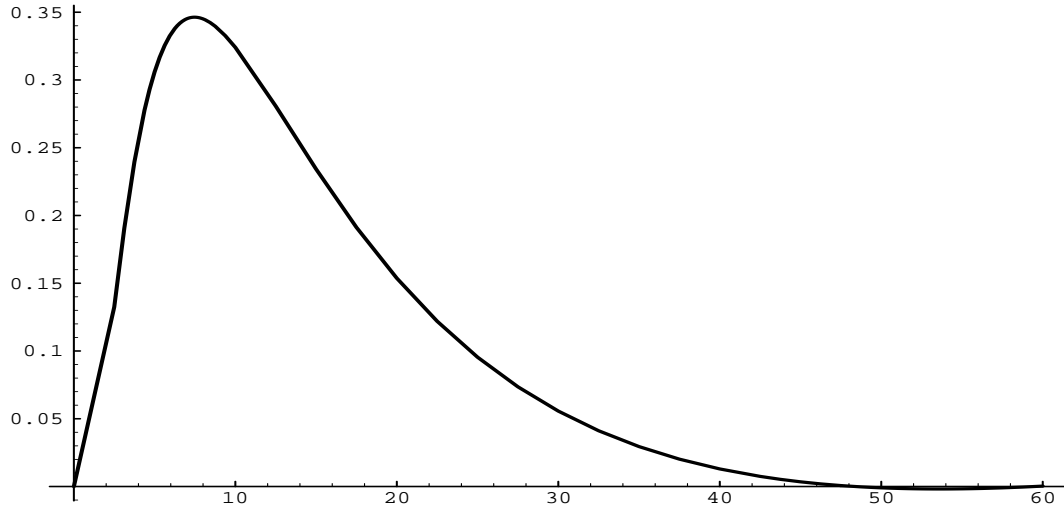FIG. 3. Free Energy as a function of z for $N_c = 5$, $p = 1.8$, $J = .32$

FIG. 4. Free Energy as a function of z for $N_c = 6$, $p = 2$, $J = 1/3$